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# TOROIDAL SURGERIES AND THE GENUS OF A KNOT

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## Abstract

In this paper we give an upper bound for the slopes yielding an incompressible torus by surgery on a hyperbolic knot in the 3-sphere in terms of its genus.

## 1. Introduction

One way to get closed 3-manifolds from a knot in  $S^3$  is the so-called *Dehn surgery*, which consists of removing a regular neighborhood  $\mathcal{N}(K)$  of the knot  $K$  and fill it with a solid torus glued differently.

The different ways of doing surgery on a knot  $K$  are parametrized by *slopes*, that is, isotopy classes of essential closed simple curves in  $\partial\mathcal{N}(K)$ , which can be identified with  $\mathbb{Q} \cup \{1/0\}$  ([25]). Given a slope  $r$  on  $\partial\mathcal{N}(K)$ , we use  $K(r)$  to denote the result of *r-Dehn surgery* on  $K$ , that is, the 3-manifold obtained by gluing a solid torus  $J$  to the exterior of  $K$ ,  $E(K) = S^3 - \text{int}\mathcal{N}(K)$ , in such a way that a meridian of  $J$  is identified with a curve of slope  $r$ .

It is known that any closed and orientable 3-manifold can be obtained by means of Dehn surgery in a link in  $S^3$  ([33]) and ([18]).

By Thurston [32], a knot  $K$  in  $S^3$  is either *hyperbolic* (i.e. its complement admits a complete Riemannian metric of constant sectional curvature  $-1$ ), *satellite* (i.e. its exterior contains an incompressible and not  $\partial$ -parallel 2-torus), or it is a *torus knot* (i.e. its exterior is Seifert fibered). By Thurston [32], if  $K$  is hyperbolic then  $K(r)$  will be hyperbolic for all but finitely many slopes  $r$ . The slopes that produce non-hyperbolic manifolds, whereas  $K$  is hyperbolic, are called *exceptional slopes*. According to Perelman [22, 23, 24] and Thurston [32], if a closed orientable 3-manifold is non-hyperbolic then it is either *reducible* (i.e. it contains an essential 2-sphere), *toroidal* (i.e. it contains an incompressible 2-torus), or it is Seifert fibered.

Many works in low dimensional topology are focused on exceptional slopes; see [6, 7, 8, 9]. In particular, if  $K$  is a hyperbolic knot, then  $K(r)$  contains an incompressible torus for only finitely many slopes  $r$  ([10] or [32]). Such slopes  $r$  and the corresponding surgeries are said to be *toroidal*. Gordon and Luecke [11] proved that any toroidal slope  $r = p/q$  is either

integral or half-integral, that is, either  $|q| = 1$  or  $2$ . On the other hand, Eudave-Muñoz [2] constructed an infinite family of hyperbolic knots  $K(l, m, n, p)$  having half-integral toroidal surgeries. Surprisingly, Gordon and Luecke proved that if  $K$  admits a half-integral toroidal surgery, then  $K$  is one of Eudave-Muñoz knots  $K(l, m, n, p)$  [12].

Teragaito obtained several results concerning integral toroidal surgeries. He showed that every integer is a toroidal surgery slope for some hyperbolic knot [26]. He also showed that any two integral toroidal slopes  $r$  and  $s$  for a hyperbolic knot satisfy  $|r - s| \leq 4$  unless  $K$  is the figure eight knot [28]. Furthermore, Gordon and Wu have determined all hyperbolic knots which have integral toroidal slopes  $r, s$ , with  $|r - s| = 4$ .

The hitting number of a toroidal surgery  $K(r)$ , denoted by  $t$ , is the minimal intersection number between a core of the attached solid torus and all incompressible tori in  $K(r)$ . For half-integral surgery, it is known that always  $t = 2$ . For integral surgery many examples have been constructed such that  $t = 4$ ; see [3], [5], [29], [30]. It is also known that there are infinitely many knots with  $t \geq 6$ ; indeed in [21], Osoinach gave an infinite family of hyperbolic knots, with a toroidal surgery, for which there is no upper bound for the hitting number  $t$ . A precise determination of the hitting number  $t$  for these examples is given in [31].

Let  $g(K)$  denote the genus of the knot  $K$ . In 2003 Teragaito proposed the following conjecture in [27].

**Conjecture 1.** *If a hyperbolic knot  $K$  in  $S^3$  has a toroidal slope  $r$ , then  $|r| \leq 4g(K)$ .*

It follows from the work of Ichihara [15] that  $|r| \leq 3 \cdot 2^{7/4}g(K)$ . Teragaito proved that his conjecture is true for genus one knots and alternating knots. On the other hand, S. Lee [17] proved that this conjecture is also true for genus two knots. For the case when  $K$  is hyperbolic and  $K(r)$  contains a Klein bottle, which in many cases is toroidal, Ichihara and Teragaito [16] proved that  $|r| \leq 4g(K)$ .

The goal of this paper is to give upper bounds for toroidal slopes close to the conjectured by Teragaito, we prove the following

**Theorem 1.** *If a hyperbolic knot  $K$  in  $S^3$  has a toroidal slope  $r$ , then  $|r| \leq 4g(K) - \frac{3}{2}$  if  $r$  is half-integral,  $|r| \leq 4g(K)$  if  $r$  is integral and  $t \geq 6$ ,  $|r| \leq 6g(K) - 3$  if  $r$  is integral and  $t = 4$  and  $|r| \leq 4g(K) + 8$  if  $r$  is integral and  $t = 2$ .*

**EXAMPLES.** The pretzel knot  $K(-2, 3, 7)$  has genus five and its set of toroidal slopes is  $\{16, 37/2, 20\}$ . For  $r = 37/2$ , we may note that the upper bound is reached in this case.

If  $K$  is the figure eight knot, its genus is one and its set of toroidal-slopes is  $\{-4, 0, 4\}$ . But  $-4$  and  $4$  are also the slopes yielding Klein bottles, and the complement of  $K$  contains a once-punctured Klein bottle.

To prove Theorem 1 we proceed as follows. By [11] we know that the toroidal slope  $r$  is integral or half-integral. In section 2 we prove the Theorem when the slope is half-integral; in this case we know all the knots with a toroidal surgery, and furthermore the genus and toroidal slopes of such knots can be calculated, so this gives a way to verify the given bound. In section 3 we develop the case when the toroidal slope is integral, this is done by means of graphs of intersection.

## 2. Half-integral slope.

Let  $K$  be a hyperbolic knot with a half-integral toroidal slope  $r$ . By [12],  $K$  is an Eudave-Muñoz knot  $K(l, m, n, p)$ , a family of knots parametrized by four integers  $l, m, n, p$  (where at least one of  $p$  and  $n$  is zero). Excluding the cases when  $l = 0, \pm 1$ ,  $m = 0$ ,  $(l, m) = \{(2, 1), (-2, -1)\}$ ,  $(m, n) = \{(1, 0), (-1, 1)\}$  and  $(l, m, p) = (2, 2, 1)$ , the knots result to be hyperbolic with only one half-integral toroidal slope. By [4], this slope is given by

$$r = \begin{cases} l(2m-1)(1-lm) + n(2lm-1)^2 - 1/2, & \text{for } K(l, m, n, 0) \text{ knots} \\ l(2m-1)(1-lm) + p(2lm-l-1)^2 - 1/2, & \text{for } K(l, m, 0, p) \text{ knots} \end{cases}$$

The mirror image of the knot  $K(l, m, n, 0)$  is the knot  $K(-l, -m, 1-n, 0)$ , and the mirror image of  $K(l, m, 0, p)$  is the knot  $K(-l, 1-m, 0, 1-p)$  [4]. So, in what follows there is no loss of generality in assuming that  $l > 0$ .

In Figures 12, 13 of [4], an explicit presentation of the knots  $K(l, m, n, p)$  as closed braids is given. However, Yi Ni observed that there is a discrepancy in the parameters in Figure 13 of [4]. Correct presentations of the knots are given in Figure 2 of [20]. In Fig. 1 we give a presentation of the knots in the case  $l > 0, n \leq 0, p \leq 0$  (which is Figure 2 of [20]), and in Fig. 2 we give a presentation of the knots in the case  $l > 0, n > 0, p > 0$ .

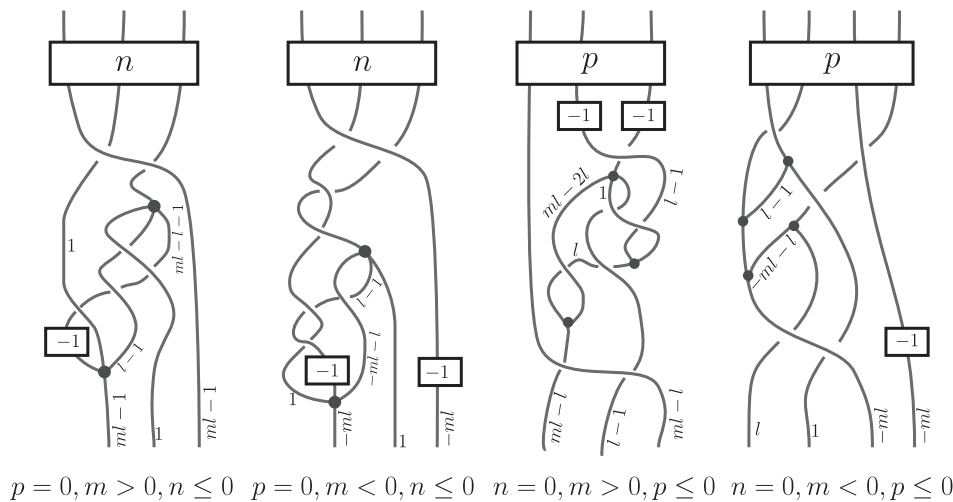


Fig. 1

It follows from this presentation that  $K(l, m, n, p)$  is the closure of a positive or negative braid, hence it is fibered, and the genus can be computed by the formula  $(C - N + 1)/2$ , where  $C$  is the crossing number in the positive or negative braid and  $N$  is the index [4].

The value of  $N$  is given by

$$N = \begin{cases} 2lm - 1, & \text{if } l > 0, m > 0 \text{ and } p = 0. \\ -2lm + 1, & \text{if } l > 0, m < 0 \text{ and } p = 0. \\ 2lm - l - 1, & \text{if } l > 0, m > 0 \text{ and } n = 0. \\ -2lm + l + 1, & \text{if } l > 0, m < 0 \text{ and } n = 0. \end{cases}$$

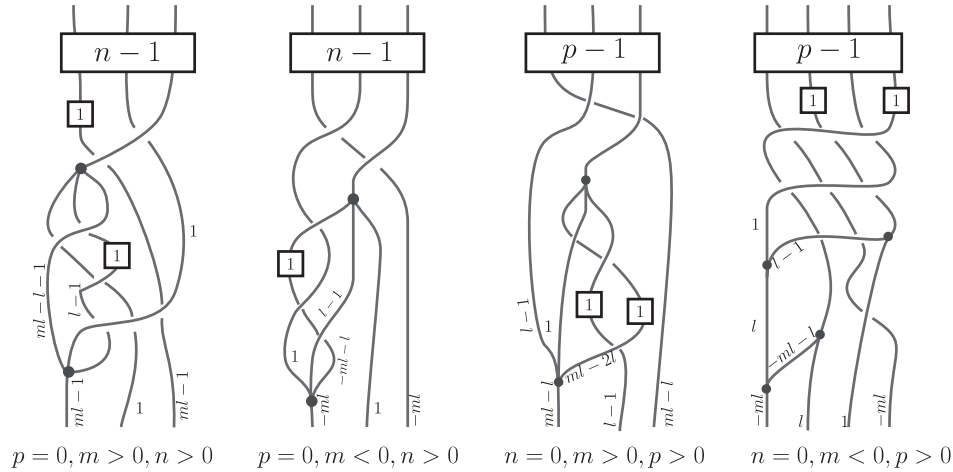


Fig.2

The explicit value of  $C$  is given in the following table (“ $>$ ” means  $> 0$ , and the analogous for  $<, \leq, \geq$ ).

$l$	$m$	$n$	$p$	Crossing number $C$
$>$	$>$	$\leq$	$0$	$2l^2m^2 - l^2m - 3lm + 2l - n(2lm - 1)(2lm - 2)$
$>$	$>$	$>$	$0$	$2l^2m^2 + l^2m - 3lm - 2l + 2 + (n - 1)(2lm - 1)(2lm - 2)$
$>$	$<$	$\leq$	$0$	$2l^2m^2 - l^2m - lm - n(2lm)(2lm - 1)$
$>$	$<$	$>$	$0$	$2l^2m^2 + l^2m - lm + (n - 1)(2lm)(2lm - 1)$
$>$	$>$	$0$	$\leq$	$2l^2m^2 - l^2m - 3lm + l - p(2lm - l - 1)(2lm - l - 2)$
$>$	$>$	$0$	$>$	$2l^2m^2 - 3l^2m - 3lm + l^2 + 2l + 2 + (p - 1)(2lm - l - 1)(2lm - l - 2)$
$>$	$<$	$0$	$\leq$	$2l^2m^2 - l^2m - lm + l - p(2lm - l - 1)(2lm - l)$
$>$	$<$	$0$	$>$	$2l^2m^2 - 3l^2m - lm + l^2 + (p - 1)(2lm - l - 1)(2lm - l)$

Computing the genus  $g$  of  $K(l, m, n, p)$  by the formula  $(C - N + 1)/2$  we have,

$l$	$m$	$n$	$p$	Genus $g$
$>$	$>$	$\leq$	$0$	$1/2[2l^2m^2 - l^2m - 5lm + 2l + 2 - n(2lm - 1)(2lm - 2)]$
$>$	$>$	$>$	$0$	$1/2[2l^2m^2 + l^2m - 5lm - 2l + 4 + (n - 1)(2lm - 1)(2lm - 2)]$
$>$	$<$	$\leq$	$0$	$1/2[2l^2m^2 - l^2m + lm - n(2lm)(2lm - 1)]$
$>$	$<$	$>$	$0$	$1/2[2l^2m^2 + l^2m + lm + (n - 1)(2lm)(2lm - 1)]$
$>$	$>$	$0$	$\leq$	$1/2[2l^2m^2 - l^2m - 5lm + 2l + 2 - p(2lm - l - 1)(2lm - l - 2)]$
$>$	$>$	$0$	$>$	$\frac{1}{2}[2l^2m^2 - 3l^2m - 5lm + l^2 + 3l + 4 + (p - 1)(2lm - l - 1)(2lm - l - 2)]$
$>$	$<$	$0$	$\leq$	$1/2[2l^2m^2 - l^2m + lm - p(2lm - l - 1)(2lm - l)]$
$>$	$<$	$0$	$>$	$1/2[2l^2m^2 - 3l^2m + lm + l^2 - l + (p - 1)(2lm - l - 1)(2lm - l)]$

We will see that  $4g - |r| \geq 3/2$  for each case.

CASE 1.  $l, m > 0, n \leq 0$  and  $p = 0$ .

Note that if  $l, m > 0$  and  $n \leq 0$  then  $|r| = -r$ .

Therefore  $4g - |r| = lm(l(2m - 1) - 8) + 3l + n(-4l^2m^2 + 8lm - 3) + 7/2$ .

Suposse  $n = 0$ , then  $4g - |r| = lm(l(2m - 1) - 8) + 3l + 7/2$ . Since  $(m, n) \neq (1, 0)$  and  $l \neq \pm 1$ , then  $l, m \geq 2$ . If  $l = m = 2$ , then  $4g - |r| = lm(l(2m - 1) - 8) + 3l + 7/2 = 2l(3l - 8) + 3l + 7/2 = 3/2$ . When  $l \geq 3$  and  $m \geq 2$ ,  $4g - |r| = lm(l(2m - 1) - 8) + 3l + 7/2 \geq lm + 3l + 7/2 \geq 37/2$ . When  $m \geq 3$  and  $l \geq 2$ ,  $4g - |r| = lm(l(2m - 1) - 8) + 3l + 7/2 \geq 2lm + 3l + 7/2 \geq 43/2$ .

Now we can assume that  $n \leq -1$  and  $l \geq 2$ . Then  $4g - |r| = lm(l(2m - 1) - 8) + 3l + n(-4l^2m^2 + 8lm - 3) + 7/2 \geq lm(l(2m - 1) - 8) + 3l + 4l^2m^2 - 8lm + 3 + 7/2 = 6l^2m^2 - l^2m - 16lm + 3l + 13/2 = lm(l(6m - 1) - 16) + 3l + 13/2$ . If  $m = 1$ , then  $l \geq 3$  (since  $(l, m) \neq (2, 1)$ ), then  $4g - |r| \geq lm(l(6m - 1) - 16) + 3l + 13/2 \geq -l + 3l + 13/2 \geq 25/2$ . If  $m \geq 2$ , then  $4g - |r| \geq lm(l(6m - 1) - 16) + 3l + 13/2 \geq 12l + 3l + 13/2 \geq 43/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 2.  $l, m, n > 0$  and  $p = 0$ .

Here  $|r| = r$ . Also, we have  $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + (n - 1)(4l^2m^2 - 8lm + 3)$ . Note that  $lm \geq 3$  since  $l \neq \pm 1$  and  $(l, m) \neq (2, 1)$ .

If  $n = 1$ , then  $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 = 2(lm - 2)^2 + l(lm - 3) - 1/2 \geq 3/2$ .

Now suposse that  $n \geq 2$  and  $l \geq 2$ . Then  $4g - |r| = 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + (n - 1)(4l^2m^2 - 8lm + 3) \geq 2l^2m^2 + l^2m - 8lm - 3l + 15/2 + 4l^2m^2 - 8lm + 3 = 6l^2m^2 + l^2m - 16lm - 3l + 21/2 = 2l^2m^2 + 4(lm - 2)^2 + l(lm - 3) - 11/2 \geq 18 + 4 - 11/2 = 33/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 3.  $l > 0, m < 0, n \leq 0$  and  $p = 0$ .

Here  $|r| = -r$ . Also we have,  $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - n(4l^2m^2 - 1) - 1/2$ . Note that  $lm \leq -2$  since  $l \geq 2$  and  $m \leq -1$ .

Suposse  $n = 0$ , then  $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 = 2(lm + 1)^2 + l(-lm - 1) - 5/2 \geq 2 + 2 - 5/2 = 3/2$ .

Now we can assume that  $n \leq -1$ , then  $4g - |r| \geq 2l^2m^2 - l^2m + 4lm - l + 4l^2m^2 - 1 - 1/2 = 6l^2m^2 - l^2m + 4lm - l - 3/2 = 4l^2m^2 + 2(lm + 1)^2 + l(-lm - 1) - 7/2 \geq 16 + 2 + 2 - 7/2 = 33/2 > 3/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 4.  $l > 0, m < 0, n > 0$  and  $p = 0$ .

Since  $n \geq 1$ , then  $r \geq l(lm - 1)(2m + 1) + 1/2 \geq 0$ , so  $|r| = r$ . Therefore  $4g - |r| = 2l^2m^2 + l^2m + 4lm + l - 1/2 + (n - 1)(4l^2m^2 - 1)$ .

Suposse  $n = 1$ , then  $m \leq -2$  (since  $(m, n) \neq (-1, 1)$ ) and  $4g - |r| = 2l^2m^2 + l^2m + 4lm + l - 1/2 \geq (3l - 4)(2l) + l - 1/2 \geq 19/2$ .

Now, if  $n \geq 2$ , then  $4g - |r| \geq 2l^2m^2 + l^2m + 4lm + l - 1/2 + 4l^2m^2 - 1 = lm(l(6m + 1) + 4) + l - 3/2 \geq 25/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 5.  $l > 0, m > 0, n = 0$  and  $p \leq 0$ .

Here  $|r| = -r$ , then  $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2 - p(2lm - l - 1)(2lm - l - 3)$ . Note that  $l, m \geq 2$  since  $l \neq 1$  and  $(m, n) \neq (1, 0)$ .

Suposse  $p = 0$ , then  $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2$ . If  $m = 2$ , then  $4g - |r| = (3l - 2)(2l - 3) - 5/2 \geq 4 - 5/2 = 3/2$ . If  $m \geq 3$ , then  $4g - |r| = lm(l(2m - 1) - 8) + 3l + 7/2 \geq 12 + 6 + 7/2 = 43/2$ .

Now we can assume that  $p \leq -1$ . Then  $4g - |r| = 2l^2m^2 - l^2m - 8lm + 3l + 7/2 - p(2lm - l - 1)(2lm - l - 3) \geq 2l^2m^2 - l^2m - 8lm + 3l + 7/2 + 4l^2m^2 - 2l^2m - 6lm - 2l^2m + l^2 + 3l - 2lm + l + 3 = lm(l(6m - 5) - 16) + l^2 + 7l + 13/2$ . If  $m = 2$ , then  $4g - |r| \geq 2l(7l - 16) + l^2 + 7l + 13/2 = 15l^2 - 25l + 13/2 = 5l(3l - 5) + 13/2 \geq 33/2$ . If  $m \geq 3$ , then  $4g - |r| \geq lm(l(6m - 5) - 16) + l^2 + 7l + 13/2 \geq 60 + 4 + 14 + 13/2 > 3/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 6.  $l > 0, m > 0, n = 0$  and  $p > 0$ .

In this case  $|r| = r$ , then  $4g - |r| = 2l^2m^2 - 3l^2m - 8lm + l^2 + 5l + 15/2 + (p - 1)(2lm - l - 1)(2lm - l - 3)$ . Note also that  $l, m \geq 2$  since  $l \neq 1$  and  $(m, n) \neq (1, 0)$ .

Suposse  $p = 1$ . Since  $(l, m, p) \neq (2, 2, 1)$ , then  $l \geq 3$  or  $m \geq 3$ . If  $m = 2$ , then  $l \geq 3$  and  $4g - |r| = (3l - 2)(l - 3) + 3/2 \geq 3/2$ . If  $l = 2$ , then  $m \geq 3$  and  $4g - |r| = 4m(2m - 7) + 14 + 15/2 \geq 2 + 15/2 > 3/2$ . Now, if  $l \geq 3$  and  $m \geq 3$ , then  $4g - |r| = lm(l(2m - 3) - 8) + l^2 + 5l + 15/2 \geq 9 + 9 + 15 + 15/2 > 3/2$ .

Now if  $p \geq 2$ , then  $4g - |r| \geq 2l^2m^2 - 3l^2m - 8lm + l^2 + 5l + 15/2 + (2lm - l - 1)(2lm - l - 3) = 6l^2m^2 - 7l^2m - 16lm + 2l^2 + 9l + 21/2$ . If  $m = 2$ , then  $4g - |r| \geq (3l - 2)(4l - 5) + 1/2 \geq 12 + 1/2 > 3/2$ . If  $m \geq 3$ , then  $4g - |r| \geq lm(l(6m - 7) - 16) + 2l^2 + 9l + 21/2 \geq 36 + 8 + 18 + 21/2 > 3/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 7.  $l > 0, m < 0, n = 0$  and  $p \leq 0$ .

Here  $|r| = -r$  and  $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 - p(2lm - l - 1)(2lm - l + 1)$ .

Suppose that  $p = 0$ , then  $4g - |r| = 2l^2m^2 - l^2m + 4lm - l - 1/2 = l(m(l(2m - 1) + 4) - 1) - 1/2$ . If  $l = 2$  and  $m = -1$ , then  $4g - |r| = 3/2$ . If  $l \geq 2$  and  $m \leq -2$  then  $l(2m - 1) + 4 \leq -6$  implies that  $l(m(l(2m - 1) + 4) - 1) \geq 22$ , so  $4g - |r| \geq 43/2$ . If  $l \geq 3$  and  $m \leq -1$  then  $l(2m - 1) + 4 \leq -5$  implies that  $l(m(l(2m - 1) + 4) - 1) \geq 12$ , so  $4g - |r| \geq 23/2$ .

Now if  $p \leq -1$ , since  $l \geq 2$  and  $m \leq -1$ , then  $4g - |r| \geq 6l^2m^2 - 5l^2m + 4lm - l + l^2 - 3/2 = 6(lm + 1/3)^2 + (l - 1/2)^2 - 5l^2m - 29/12 \geq 26/3 + 9/4 + 20 - 29/12 > 3/2$ .

In both cases,  $4g - |r| \geq 3/2$ .

CASE 8.  $l > 0, m < 0, n = 0$  and  $p > 0$ .

Here  $|r| = r$  and  $4g - |r| = 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 + (p - 1)(2lm - l - 1)(2lm - l + 1)$ .

First suppose that  $p = 1$ , then  $4g - |r| = 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 = 2(lm + 1)^2 - 3l^2m + (l - 3/2)^2 - 1/2 - 9/4 - 2 \geq 2 + 12 + 1/4 - 1/2 - 9/4 - 2 = 19/2 > 3/2$ .

Now assume that  $p \geq 2$ , then  $4g - |r| \geq 2l^2m^2 - 3l^2m + 4lm + l^2 - 3l - 1/2 + (2lm - l - 1)(2lm - l + 1) = 6l^2m^2 - 7l^2m + 4lm + 2l^2 - 3l - 3/2 = 4l^2m^2 - 7l^2m + 2(lm + 1)^2 + (2l - 1)(l - 1) - 9/2 \geq 16 + 28 + 2 + 3 - 9/2 > 3/2$ .

In all cases,  $4g - |r| \geq 3/2$ .

□

### 3. Integral slope.

Let  $r$  be an integral slope and  $\widehat{T}$  an incompressible torus in  $K(r)$  that intersects the attached solid torus  $J$  in a disjoint union of meridional disks  $v_1, v_2, \dots, v_t$  numbered successively along  $J$ . We assume that  $\widehat{T}$  is chosen so that  $t$  is minimal among all incompressible tori in  $K(r)$ . We also assume that  $r$  is not the longitudinal slope, then  $\widehat{T}$  must be separating in  $K(r)$  and hence  $t$  is even.

Let  $S$  be a minimal genus  $g(K)$ , Seifert surface for  $K$ . By shrinking  $S$  suitably, we may assume that  $S$  is properly embedded in  $E(K) = S^3 - \text{int}\mathcal{N}(K)$ . We cap off  $\partial S$  with a disk  $u$  to obtain a closed surface  $\widehat{S}$ . Let  $T = \widehat{T} \cap E(K)$ . We isotope  $T$  so that  $S \cap T$  consists of circles and arcs that are essential in both  $S$  and  $T$ .

The intersection  $S \cap T$  defines two labeled graphs  $G_S$  on  $\widehat{S}$  and  $G_T$  on  $\widehat{T}$  as follows. The graph  $G_S$  has only one (fat) vertex  $u$ , while the graph  $G_T$  has  $t$  (fat) vertices  $v_1, v_2, \dots, v_t$ . For each graph  $G_S$  and  $G_T$ , the edges are the arc components of  $S \cap T$ . For each  $x = 1, 2, \dots, t$ , there are  $r$  points in  $\partial u \cap \partial v_x$ , which are endpoints of some edges in  $G_S$  ( $G_T$ ). We label these  $r$  points by  $x$  in  $G_S$ . Then labels  $1, 2, \dots, t$  appear in order around the vertex of  $G_S$



repeatedly  $r$  times. We number consecutively each of these  $r$  blocks of labels according to the orientation of  $\partial u$ , starting at some block (see Fig. 3). For each edge endpoint in  $G_S$ , we will make use of another label which is the number of the block that it belongs. Each edge of  $G_S$  has some labels  $x$  and  $y$  at its endpoints, and this edge connects  $v_x$  and  $v_y$  in  $G_T$  with labels  $i$  and  $j$ , where  $i$  and  $j$  are the labels of the blocks to which the edge endpoints belongs in  $G_S$ . Here  $x$  and  $y$  have opposite parities by the parity rule [1] and then  $x \neq y$ . Note that the number of the edges in  $G_S$  (or  $G_T$ ) is  $|r|t/2$ .

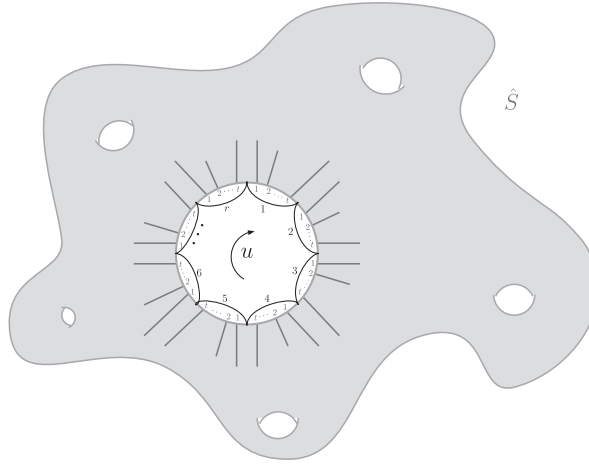


Fig. 3

Let  $\widehat{B}, \widehat{W}$  be the two sides of  $\widehat{T}$  in  $K(r)$ , also let  $B = \widehat{B} \cap E(K)$  and  $W = \widehat{W} \cap E(K)$ . For each face  $f$  of  $G_S$ , we color  $f$  black or white according to whether a collar of  $\partial f$  lies in  $B$  or  $W$ .

An edge of  $G_S$  is called  $x$ -edge if it has label  $x$  at one endpoint, and it is called an  $(x, y)$ -edge if it has label  $y$  at the other endpoint. Note that the number of  $x$ -edges in  $G_S$  is  $|r|$  for each  $x = 1, 2, \dots, t$ .

If the subgraph of  $G_S$  consisting of all  $x$ -edges contains disk faces, we call them  $x$ -faces. We frequently regard an  $x$ -face as a configuration in  $G_S$ . If an  $x$ -face is a disk face of  $G_S$ , then all the edges in the boundary of the  $x$ -face have the same label pair  $\{x, y\}$ , where  $|x - y| = 1$ . The cycle of the edges of such an  $x$ -face is called a *Scharlemann cycle*. A Scharlemann cycle with only two edges is called an *S-cycle*. A cycle of  $G_S$  immediately surrounding a Scharlemann cycle is called an *extended Scharlemann cycle*.

**Lemma 1.** *If  $|r| > 4g(K)$ ,  $K(r)$  does not contain a Klein bottle.*

Proof. This follows from [16], Corollary 1.3. □

**Lemma 2.** *If  $|r| > 2g(K) - 1$ ,  $K(r)$  is irreducible.*

Proof. Lemma 2.3 of [17] or Theorem 1.3 of [19].  $\square$

**Lemma 3.**  *$G_S$  cannot contain an extended Scharlemann cycle if  $t \geq 4$ .*

Proof. This is Theorem 3.2 of [11].  $\square$

**Lemma 4.** *No two edges are parallel in both graphs  $G_S$  and  $G_T$ .*

Proof. Lemma 2.1 of [10].  $\square$

**Lemma 5.** *If  $|r| > 4g(K)$  and  $t \geq 4$ , then  $G_S$  cannot contain two  $S$ -cycles on disjoint label pairs.*

Proof. Lemma 2.6 of [17].  $\square$

**Lemma 6.** *There are at most four labels of Scharlemann cycles in  $G_S$ .*

Proof. This is Lemma 2.3(4) of [14].  $\square$

**Lemma 7.** *The edges of a Scharlemann cycle of  $G_S$  cannot lie in a disk in  $\hat{T}$ .*

Proof. This is Lemma 3.1 of [11].  $\square$

Let  $x$  be a label of  $G_S$ . Define  $\Gamma_x$  be the subgraph of  $G_S$  consisting of all  $x$ -edges and  $\overline{F}_x$  be the number of disk faces of  $\Gamma_x$ . Thus  $\Gamma_x$  has exactly  $|r|$  edges.

**Lemma 8.** *If  $|r| \geq 4g(K) - 1$ , then  $\Gamma_x$  contains a disk face of length at most 3 for any label  $x$ .*

Proof. Assume that  $\Gamma_x$  has no disk face of length at most 3. Then  $4\overline{F}_x \leq 2|r|$ . By an Euler characteristic count in  $\hat{S}$  we have

$$\frac{|r|}{2} \geq \overline{F}_x \geq 1 - 2g(K) + |r|$$

Thus  $|r| \leq 4g(K) - 2$ , a contradiction.  $\square$

**3.1.  $t \geq 6$ .** Suppose  $|r| > 4g(K)$ . By Lemma 1,  $K(r)$  does not contain a Klein bottle. We mainly follow the argument in [[11], Section 5]. Let  $C_{\pm}(i)$  be the configuration in  $G_S$  as illustrated in Fig. 4.

Let  $S$  denote the set of labels of Scharlemann cycles of length at most 3 in  $G_S$ , and let  $\mathcal{L}_0 = \{1, 2, \dots, t\} \setminus S$ . Recall that  $\Gamma_x$  contains a disk face of length at most three for any label  $x$  by Lemma 8.

**Lemma 9.**  *$G_S$  contains a configuration  $C_{\pm}(i)$  for just one label  $i$ .*

Proof. Let  $x$  be a label in  $\mathcal{L}_0$ . Then  $\Gamma_x$  contains a bigon or a trigon face  $f$  by Lemma 8. Since there is no extended Scharlemann cycle,  $\Gamma_x$  has a disk face as shown in Figure 5.1 of [11] (see Lemma 5.1 of [11]), and hence has a configuration  $C_{\pm}(i)$ . The uniqueness of such a configuration follows from [[11], Lemma 5.4].  $\square$

**Lemma 10.**  *$|S| \leq 4$  and  $|\mathcal{L}_0| \leq 4$ .*

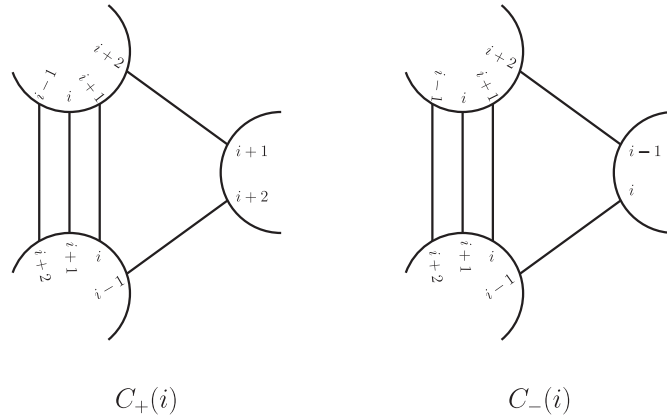


Fig.4

Proof. The first part follows from Lemma 6, and the second is Theorem 5.8 of [11].  $\square$

**Corollary 1.**  $t \leq 8$ .

Proof. By Lemma 10,  $t = |S| + |\mathcal{L}_0| \leq 8$ .  $\square$

**Proposition 1.**  $t = 6$  is impossible.

Proof. By Lemma 9, we may assume that  $G_S$  contain an  $S$ -cycle with label pair  $\{1, 2\}$ , and  $\mathcal{L}_0 \subset \{3, 4, 5, 6\}$ . By the argument of [[11], page 626], we see that  $\mathcal{L}_0 = \{3, 6\}$  and  $S = \{1, 2, 4, 5\}$ . Hence  $G_S$  contains 12- and 45-Scharlemann cycles of length 2 or 3. This is impossible by [[11], Theorem 3.9].  $\square$

**Proposition 2.**  $t = 8$  is impossible.

Proof. If  $t = 8$ , then  $|S| = |\mathcal{L}_0| = 4$ . By Lemma 9,  $G_S$  contains an  $S$ -cycle  $\rho$  with label pair  $\{i, i+1\}$  for some  $i$ , and  $\mathcal{L}_0 = \{i-2, i-1, i+2, i+3\}$  by [[11], Lemma 5.3(2)]. Since  $|S| > 2$ , there is another  $(j, j+1)$ -Scharlemann cycle  $\sigma$  of length at most 3 with  $j \neq i$ , and with  $j, j+1 \notin \mathcal{L}_0$ . Hence  $j, j+1 \notin \{i-2, i-1, i, i+1, i+2, i+3\}$ . The rest of the proof is the same as those of [[11], pages 625-626].  $\square$

**3.2.  $t = 4$ .** Suppose that  $|r| \geq 6g(K) - 2$ . By Lemmas 5 and 6, we can take a label  $x$  of  $G_S$  which is not a label of any  $S$ -cycle. Every disk face of  $\Gamma_x$  has at least 3 sides, since otherwise  $G_S$  would contain an extended Scharlemann cycle. An Euler characteristic count for  $\Gamma_x$  gives  $\overline{F}_x \geq 1 - 2g(K) + |r|$ . Also  $2|r| \geq 3\overline{F}_x$ , then  $|r| \leq 6g(K) - 3$ , a contradiction.

**3.3.  $t = 2$ .** Suppose that  $|r| \geq 4g(K) + 1$ . Since  $t = 2$ ,  $G_T$  has exactly two vertices, then has at most four edge classes as shown in Fig. 5.

We label each edge  $e$  of  $G_S$  by the label of its class on  $G_T$  and we denote this label by  $\mathcal{L}(e)$ .

Note that when  $t = 2$ , all disk faces of  $G_S$  are Scharlemann cycles.

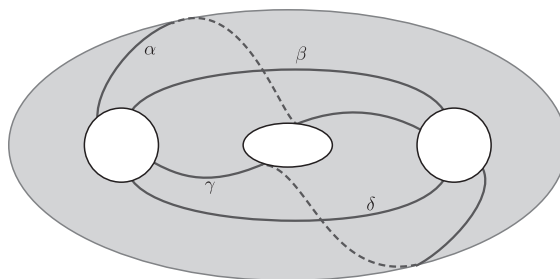


Fig. 5

Let  $f$  be a disk face of  $G_S$  and suppose that  $\chi_1, \chi_2$  are two distinct edge classes in  $G_T$ . We say that  $f$  is a  $(\chi_1, \chi_2)$ -face if the edges of  $f$  belong to  $\chi_1 \cup \chi_2$ . When  $f$  is a  $(\chi_1, \chi_2)$ -face,  $f$  is said to be  $\chi_i$ -good ( $i = 1, 2$ ) if no two consecutive edges of  $f$  belong to class  $\chi_i$ . If  $f$  is  $\chi_i$ -good for some  $i = 1, 2$ , then  $f$  is said to be  $(\chi_1, \chi_2)$ -good. We denote by  $|f|$  the number of edges of  $f$ .

**Lemma 11.** *For any two edge class labels  $\chi_1, \chi_2$ ,  $G_S$  cannot have  $(\chi_1, \chi_2)$ -good faces on both sides of  $\widehat{T}$ .*

Proof. This is Lemma 4.2 of [17]. □

In what follows, let  $H_{i,i+1}$  be the part of  $J$  between consecutive components  $i$  and  $i + 1$  of  $\partial T$ .

**Lemma 12.** *If a disk face of  $G_S$  have at most 3 edges, the edges are contained in an essential annulus on  $\widehat{T}$ .*

Proof. Let  $\sigma$  be a disk face of  $G_S$ , then  $\sigma$  is a Scharlemann cycle.

If  $|\sigma| = 2$ , the result is a direct consequence of Lemma 7.

If  $|\sigma| = 3$ . Let  $a, b, c$  be the edges of  $\sigma$  and suppose  $\mathcal{L}(a) \neq \mathcal{L}(b) \neq \mathcal{L}(c)$  (otherwise we would have finished). Extending the edges  $a, b, c$  to the corners of  $\sigma$  in  $T \cup H_{1,2}$ , they look like Figure 6. According to such a figure, to complete the cycle  $\sigma$  we need to connect the ends of the arcs  $a, b, c$  in  $H_{1,2}$ . We have two options: (i) to connect an end of  $a$  with an end of  $c$ , an end of  $b$  with an end of  $a$  and an end of  $c$  with an end of  $b$ ; or (ii) to connect an end of  $a$  with an end of  $b$ , an end of  $b$  with an end of  $c$  and an end of  $c$  with an end of  $a$ . However, it is not difficult to see that is not possible to realize such connections in  $H_{1,2}$  without obtaining autointersections. □

Note that by Lemma 12, the disk faces of  $G_S$  of length at most three are good faces.

**Lemma 13.**  *$G_S$  does not contain two bigons of the same color on distinct edge class pairs.*

Proof. Otherwise,  $K(r)$  would contain a Klein bottle. See the proof of Lemma 5.2 [13]. □

**Lemma 14.** *If a bigon and a trigon of  $G_S$  have the same color, then they have disjoint pairs of edge class labels.*

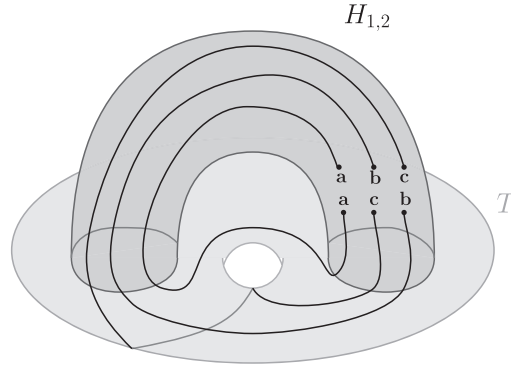


Fig. 6

Proof. This is Lemma 4.5 of [17].  $\square$

- Lemma 15.** (1) *If two trigons of  $G_S$  of the same color have different pairs of class labels, then the pairs are disjoint.*  
 (2) *Suppose that two trigons of  $G_S$  of the same color have the same pair of edge class labels, say,  $\{\chi_1, \chi_2\}$ . If one has an edge in class  $\chi_1$  and two edges in class  $\chi_2$ , then the other also has one edge in class  $\chi_1$  and two edges in class  $\chi_2$ .*  
 (3) *If two trigons of  $G_S$  have opposite colors, then they cannot have the same pair of edge class labels.*

Proof. This is Lemma 4.6 of [17]  $\square$

Recall that  $u$  is the unique vertex of  $G_S$ . Let  $a, b$  be some two intersection points of  $\partial u$  and  $\partial v_x$  ( $x = 1, 2$ ). Then both points have label  $x$  in  $G_S$ . Since  $r$  is an integral slope, the points  $a$  and  $b$  are consecutive on  $\partial v_x$  of  $G_T$  if and only if there is exactly one edge endpoint in  $G_S$  between the points. See Fig. 7, where  $x = 1$ .

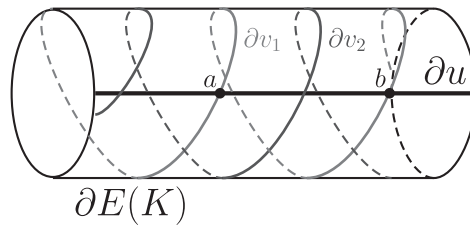


Fig. 7

We orient  $\partial v_1$  counterclockwise around  $v_1$ ,  $\partial v_2$  clockwise around  $v_2$  and  $\partial u$  counterclockwise around  $u$ . We may assume that the three curves  $\partial v_1$ ,  $\partial v_2$  and  $\partial u$  proceed in the same direction along the knot  $K$  when they proceed along their orientations.

Let  $\chi \in \{\alpha, \beta, \gamma, \delta\}$ . For two edges  $e, e'$  in class  $\chi$ , we write  $e < e'$  if the point  $e \cap \partial v_1$  precedes the point  $e' \cap \partial v_1$  with respect to the orientation of  $\partial v_1$ . Note that  $e < e'$  if and only

if the point  $e \cap \partial v_2$  precedes the point  $e' \cap \partial v_2$  with respect to the orientation of  $\partial v_2$ . We say that  $e$  is the *first edge* in the class  $\chi$  if  $e < e'$  for any other edge  $e'$  in class  $\chi$ . Similarly, the last edge is defined.

**Lemma 16.** *Let  $a_1, a_2$  be two edge endpoints of  $G_S$  such that there is exactly one endpoint between them as in Fig. 8. Let  $e_i$  be the edge of  $G_S$  incident to  $a_i$  ( $i = 1, 2$ ). Let  $\chi_i = \mathcal{L}(e_i)$  and assume  $\chi_1 \neq \chi_2$ . Then on  $G_T$ ,  $e_1$  is the last edge in class  $\chi_1$ , while  $e_2$  is the first edge in class  $\chi_2$ . Also,  $a_1, a_2$  appear consecutively and in order on a vertex of  $G_T$ .*

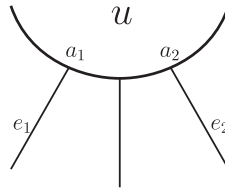


Fig. 8

Proof. This is Lemma 4.7 of [17]. □

**Lemma 17.** *Any two bigons of  $G_S$  cannot be adjacent.*

Proof. Lemma 4.8 of [17]. □

**Lemma 18.** *If a bigon and a trigon of  $G_S$  are adjacent, then  $G_S$  contains only one bigon.*

Proof. It follows from the first part of the proof of Lemma 4.9 of [17]. □

Recall that the graph  $G_S$  have  $|r|$  edges. Let  $F$  be the number of disk faces on  $G_S$ .

An Euler characteristic calculus gives  $1 - |r| + F \geq 2 - 2g$ , then

$$(1) \quad F \geq |r| - 2g(K) + 1$$

**Lemma 19.** *If  $|r| \geq 6g(K) - 2$ , then  $G_S$  cannot have a bigon and a trigon which are adjacent.*

Proof. Suppose there is a bigon adjacent to a trigon in  $G_S$ . By Lemma 18,  $G_S$  contains only one bigon. Applying (1),  $2|r| \geq 3(F - 1) + 2 = 3F - 1 \geq 3(|r| + 1 - 2g(K)) - 1 = 3|r| - 6g(K) + 2$ , hence  $|r| \leq 6g(K) - 2$ . Also  $|r| \geq 6g(K) - 2$  by assumption, then  $|r| = 6g(K) - 2$ .

Let  $F^b, F^w$  be the number of black and white disk faces of  $G_S$ , respectively. Suppose that the unique bigon on  $G_S$  is black. Then  $|r| \geq 3(F^b - 1) + 2 = 3F^b - 1$ , so  $F^b \leq (|r| + 1)/3$ . Hence  $(|r| + 1)/3 + F^w \geq F^b + F^w = F \geq |r| + 1 - 2g(K)$ , which yields  $F^w \geq (2|r| + 2)/3 - 2g(K) = 2g(K) - 2/3$  and hence,  $F^w \geq 2g$ . Also  $|r| \geq 3F^w$ , then  $|r| \geq 3F^w \geq 6g(K)$ , contradicting that  $|r| = 6g(K) - 2$ . □

**Lemma 20.** *Let  $f$  be a disk face of  $G_S$  with  $|f| \geq 4$ . Then the following hold.*

- *$f$  cannot be surrounded by bigons.*
- *If  $|f|$  is odd, then  $f$  is adjacent to at most  $|f| - 2$  bigons.*

Proof. Lemma 4.10 of [17]. □

**Lemma 21.**  *$G_S$  cannot contain a bigon, a trigon and a tetragon of the same color.*

Proof. Lemma 4.11 of [17]. □

**DEFINITION.** We say that two disk faces in  $G_S$  are *of the same type* if they have the same color, the same length, the same set of class edges labels and the same number of edges in each edge class. This is an equivalence relation in the set of disk faces of  $G_S$ , and we call as *a type of  $n$ -faces* an equivalence class of a disk face with  $n$  sides.

Note that by Lemmas 12 and 13, there are at most two types of bigons in  $G_S$  (one for each color). Also, by Lemma 15, there are at most four types of trigons in  $G_S$  (two for each color).

**Convention on the Figures.** From now on, the same number and shape of big dots on the edges in the graph  $G_S$  will indicate that the edges are the same.

**DEFINITION.** Two disk faces of the same type in  $G_S$ , are *consecutive with respect to one corner* if the corresponding labels of one of their corners are consecutive in  $G_T$ .

**REMARK.** Note that if two disk faces are consecutive with respect to one corner and if in addition the corresponding edges that form the corners are parallel edges in  $G_T$ , then the faces are consecutive with respect to the other corners at the ends of the edges.

**DEFINITION.** A finite set of disk faces in  $G_S$ ,  $\{\sigma_i\}_I$  is *consecutive* if the faces are of the same type and we can enumerate them as  $\{\sigma_j\}_{j=1}^m$  for some  $m \geq 1$ , in such a way that for all  $1 \leq j \leq m$ ,  $\sigma_j$  and  $\sigma_{j+1}$  are consecutive faces with respect to one corner (and then with respect to all their corners by the remark above).

Fig. 9 shows an example of a consecutive set of faces  $\sigma_1, \sigma_2, \dots, \sigma_m$  in  $G_S$ . The blue labels are the labels that they have in  $G_T$  and which are supposed to be consecutive.

We denote by  $F_i$  the set of faces with  $i$  sides in  $G_S$  and by  $|F_i|$  its cardinality. In the same way, we denote by  $F_i^w$  and  $F_i^b$  be the set of white and black disk faces with  $i$  sides in  $G_S$ , respectively.

**Lemma 22.** *The set of bigons of the same color in  $G_S$ , is consecutive.*

Proof. Fix a color, say white. By Lemma 13, the bigons in  $F_2^w$  have the same two edge classes, say  $\{\chi_1, \chi_2\}$ , then they are all of the same type. In fact, the edges of the bigons in  $F_2^w$  lie in an annulus  $A$  on  $\widehat{T}$ . Let  $a_i^f$  and  $b_i^l$  be the first and last edge, respectively of class  $\chi_i$ ,  $i = 1, 2$  for bigons in  $F_2^w$ . Suppose we have an edge  $c$  on a white face  $\sigma$  such that  $a_1^f < c < b_1^l$  or  $a_2^f < c < b_2^l$ . Note that when going along the edges of  $\sigma$  on  $\widehat{T} \cup H_{1,2}$  they cannot get out

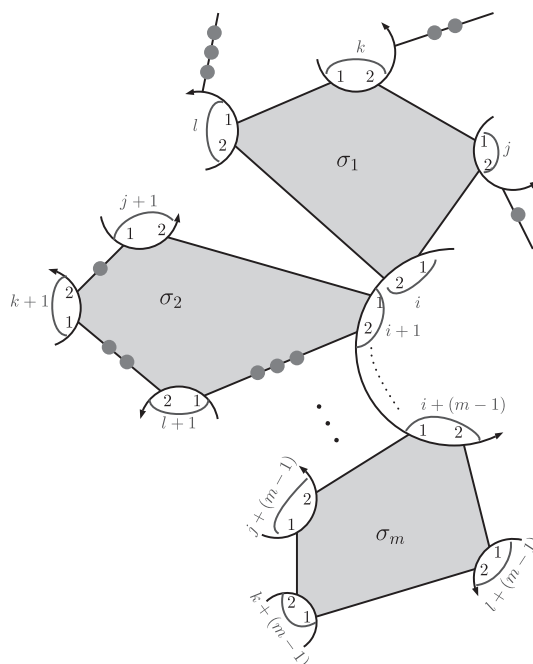


Fig.9

of  $A$  and also they cannot give more than one lap around  $A$  because if not  $\sigma$  would not close. Then  $\sigma$  is a bigon inside  $A$  with edge classes  $\{\chi_1, \chi_2\}$ . Therefore the edges of the bigons in  $F_2^w$  of the same class, are consecutive on  $G_T$  and hence the set of white bigons is consecutive in  $G_S$ .  $\square$

**Lemma 23.** *The bigons of the same color are adjacent to at most two faces in  $G_S$ , each one having the same number of edges belonging to bigons.*

*Proof.* Fix a color, say black. By Lemma 22,  $F_2^b$  is a consecutive set in  $G_S$ . Then we can enumerate its elements according to the order they appear on the vertex  $u$  of  $G_S$ . Suppose that in such an order, the bigons are  $b_m$ ,  $m \in \{1, 2, \dots, n\}$  and have labels  $i + (m - 1)$ ,  $j + (m - 1)$ , respectively on its corners on  $G_T$  for some fixed  $i, j \in \{1, 2, \dots, |r|\}$ . Start with the first bigon, it is adjacent to at most two faces,  $f_1, f_2$  (not necessarily different). By following the labels of the blocks in  $\partial u$  from  $i$  to  $i + 1$ , we note that the second bigon must be adjacent to the face  $f_2$  on one side and adjacent to the face  $f_1$  on the other side (if we follow the labels from  $j$  to  $j + 1$ ), so they appear in  $G_S$  as in Fig. 10.

The third bigon has labels  $i + 2, j + 2$  on its corners, then it must be adjacent to the face  $f_2$  by one side (if we follow the labels from  $j + 1$  to  $j + 2$ ), and adjacent to the face  $f_1$  on the other side (if we follow the labels from  $i + 1$  to  $i + 2$ ), as in Fig. 11. Continuing with this process, we get that all bigons are adjacent to the same two faces, and in fact  $f_1$  and  $f_2$  have the same number of edges belonging to bigons.  $\square$

We have similar properties for the trigons.

**Lemma 24.** *The set of trigons of the same type is consecutive in  $G_S$ .*



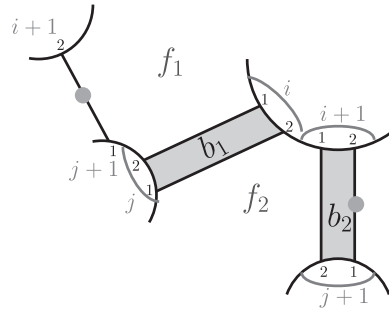


Fig. 10

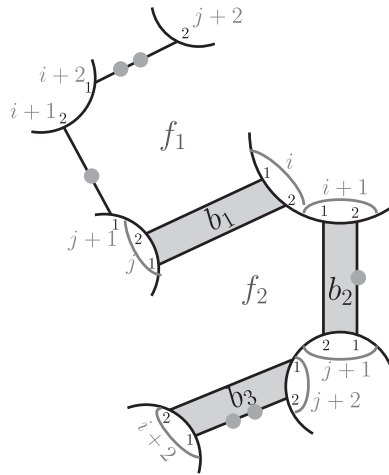


Fig. 11

**Proof.** Let  $C$  be the set of trigons of the same type in  $G_S$ . Without loss of generality, suppose that the trigons in  $C$  are white and have one edge of class  $\chi_1$  and two edges of class  $\chi_2$ . By the remark above, we just need to prove that the trigons of the same type are consecutive with respect to the corner which contain the edge of class  $\chi_1$  on each trigon. By Lemma 12, the edges of the trigons of the same type lie in an essential annulus  $A$  on  $\widehat{T}$ . Let  $a^f$  and  $b^l$  be the first and last edge of class  $\chi_1$  for trigons in  $C$ , respectively. Suppose we have an edge  $c$  on a white face  $\sigma$  such that  $a^f < c < b^l$ . Note that when going along the edges of  $\sigma$  on  $\widehat{T} \cup H_{1,2}$  they cannot get out of  $A$  and also they cannot take more than one lap around  $A$  because if not  $\sigma$  would not close (see Fig. 12). Then  $\sigma$  is a trigon inside  $A$  with edge classes  $\{\chi_1, \chi_2\}$ . Therefore the edges of the trigons must be consecutive on  $G_T$  and then the set of trigons is consecutive in  $G_S$ .  $\square$

**Lemma 25.** *The trigons of the same type are adjacent to at most three faces on  $G_S$ , each one having the same number of edges belonging to trigons.*

**Proof.** Let  $C$  be the set of trigons of the same type in  $G_S$ . By Lemma 24, the set  $C$  is consecutive. Suppose that the trigons in  $C$  have one edge of class  $\chi_1$  and two edges of class  $\chi_2$ . We can enumerate the trigons in  $C$  according to the order that the edges of class  $\chi_1$  of the trigons appear on the vertex  $u$ . Suppose that in such an order, the trigons are  $c_m$ ,

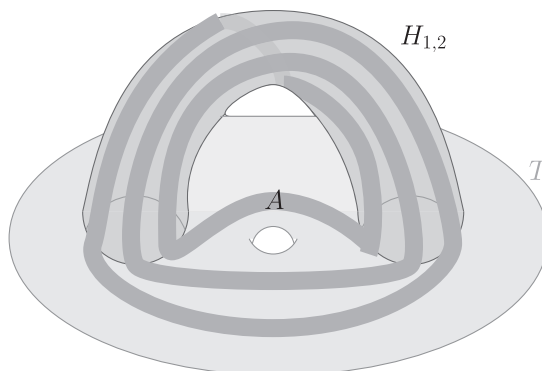


Fig. 12

$m \in \{1, 2, \dots, n\}$  and have edge labels in  $G_T$ ,  $i + (m - 1), j + (m - 1), l + (m - 1) \pmod{|r|}$ , respectively for some fixed numbers  $i, j, l \in \{1, 2, \dots, |r|\}$ . Start with the first trigon, it is adjacent to at most three faces  $f_1, f_2, f_3$  (not necessarily different). By following the labels from  $i$  to  $i + 1$  on the vertex  $u$ , we note that the second trigon is adjacent to the face  $f_2$  by one side, adjacent to the face  $f_1$  by other side and adjacent to  $f_3$  on the third side (if we follow the labels from  $j$  to  $j + 1$  and also from  $l$  to  $l + 1$ ), so they appear in  $G_S$  like in Fig. 13.

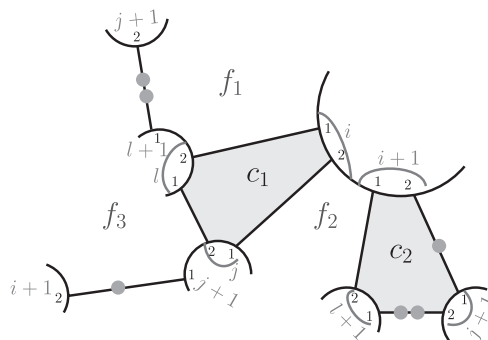


Fig. 13

The third trigon has edge labels  $i + 2, j + 2, l + 2$  in  $G_T$ . Hence it must be adjacent to  $f_2$  by one side (if we follow the labels from  $l + 1$  to  $l + 2$ ), and adjacent to  $f_1$  (if we follow the labels from  $j + 1$  to  $j + 2$ ) and  $f_3$  (if we follow the labels from  $i + 1$  to  $i + 2$ ) on its other sides, like in Fig. 14. Continuing with this process, we get that all trigons of the same type must glue to the same faces and in fact,  $f_1, f_2$  and  $f_3$  have the same number of edges belonging to trigons.  $\square$

**Lemma 26.** *If on  $G_S$  there are two adjacent trigons, they are unique with respect to their type.*

**Proof.** Suppose that we have two adjacent trigons  $c_1$  and  $c_2$  on  $G_S$  (say  $c_1$  black and  $c_2$  white). Without loss of generality, we can assume that  $c_1$  is a  $\{\chi_1, \chi_2\}$ -good face and  $c_2$  is a  $\{\chi_2, \chi_3\}$ -good face. By Lemma 25,  $c_1$  is adjacent to all white trigons of type  $\{\chi_2, \chi_3\}$ , and in the same way  $c_2$  is adjacent to all black trigons of type  $\{\chi_1, \chi_2\}$ . Since  $c_1$  and  $c_2$

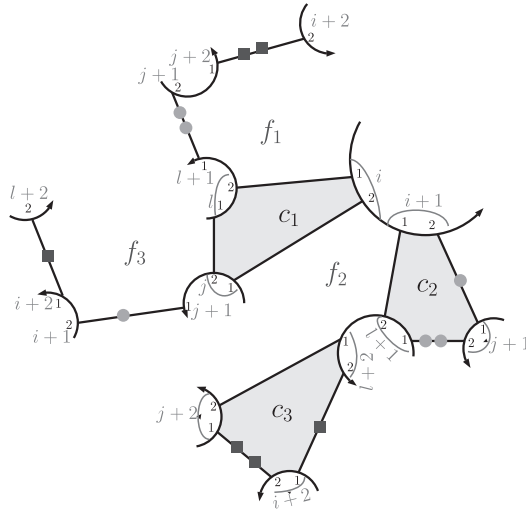


Fig. 14

are adjacent then there are at most two black trigons of type  $\{\chi_1, \chi_2\}$  and two white trigons of type  $\{\chi_2, \chi_3\}$ . Suppose  $c_1$  is adjacent to two white trigons of type  $\{\chi_2, \chi_3\}$  (otherwise we would have finished). It means that  $c_1$  have two edges of class  $\chi_2$  and one of class  $\chi_1$ . Applying Lemma 16 on the corners of  $c_1$  containing the edge of class  $\chi_1$ , we can see that this is the only edge of its class. This implies that there are no more black trigons of type  $\{\chi_1, \chi_2\}$ .

Let  $a, b, c, d$  be the rest of edges of the trigons according to Fig. 15.

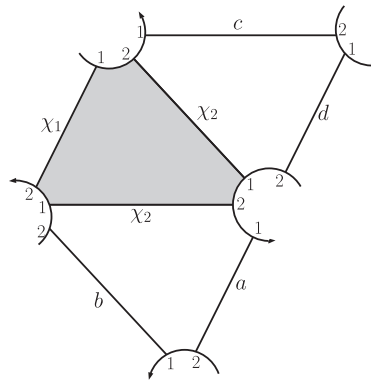


Fig. 15

Applying Lemma 16, we see that  $c$  is the first edge on its class, while  $b$  is the last edge on its class. For the edges  $a, b, c, d$  we have the following classes options. The superscripts  $f$  and  $l$  indicate that the edge is the first or the last of its class, respectively.

- (1)  $\mathcal{L}(a) = \chi_2; \mathcal{L}(b) = \chi_3^l; \mathcal{L}(c) = \chi_2^f; \mathcal{L}(d) = \chi_3$
- (2)  $\mathcal{L}(a) = \chi_2; \mathcal{L}(b) = \chi_3^l; \mathcal{L}(c) = \chi_3^f; \mathcal{L}(d) = \chi_2$
- (3)  $\mathcal{L}(a) = \chi_3; \mathcal{L}(b) = \chi_2^l; \mathcal{L}(c) = \chi_2^f; \mathcal{L}(d) = \chi_3$
- (4)  $\mathcal{L}(a) = \chi_3; \mathcal{L}(b) = \chi_2^l; \mathcal{L}(c) = \chi_3^f; \mathcal{L}(d) = \chi_2$

Applying Lemma 16 on the first case, we have that  $d$  is the last edge of the class  $\chi_3$ , which contradicts the fact that  $b$  is such an edge.

On the second case, since the edges of the class  $\chi_2$  are consecutive, we can label them with  $i, j$ , like in Fig. 16. However such configuration implies that  $b$  and  $c$  are the same, which is not possible since we are assuming that they lie in different faces.

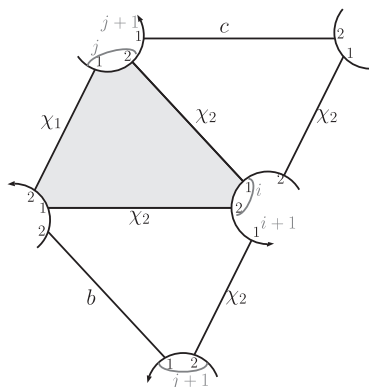


Fig. 16

Applying Lemma 16 on the third case, we have that the edges of class  $\chi_2$  in the black trigon are the first and the last edges of its class, contradicting that  $b$  and  $c$  are such edges.

Applying Lemma 16 on the fourth case, we see that one of the edges of class  $\chi_2$  is the last edge of this class, contradicting that  $b$  is such an edge.

Finally, applying Lemma 16 on the last case, we see that  $a$  and  $d$  are the first and last edge of class  $\chi_3$ , respectively, contradicting that  $c$  and  $b$  are such edges.  $\square$

Note that Lemmas 15 and 26 imply that on  $G_S$  there are at most two pairs of adjacent trigons and in such case the trigons are unique.

Denote by  $n$  the number of faces to which the bigons (of both colors) in  $G_S$  are adjacent and denote by  $m$  the number of faces to which the trigons (of both colors) in  $G_S$  are adjacent. By Lemmas 23 and 25,  $n \leq 4$  and  $m \leq 12$ , respectively.

We use Lemmas from 11 to 26 to separate the proof of the theorem on eleven cases. In Fig. 17 we see the tree of cases and the bound obtained for  $|r|$  on each case.

By counting the edges on  $G_S$  we have that  $2|r| \geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|) = 4F - 2|F_2| - |F_3|$ . By (1),  $2|r| \geq 4(|r| - 2g(K) + 1) - 2|F_2| - |F_3|$ . Since  $|F_2| = |F_3| = 0$ , then  $|r| \leq 4g(K) - 2$ , which finishes this case.

Since any two bigons cannot be adjacent by Lemma 17 and  $G_S$  does not have any trigon, then the faces to which the bigons are adjacent have at least four sides. On the other hand, a

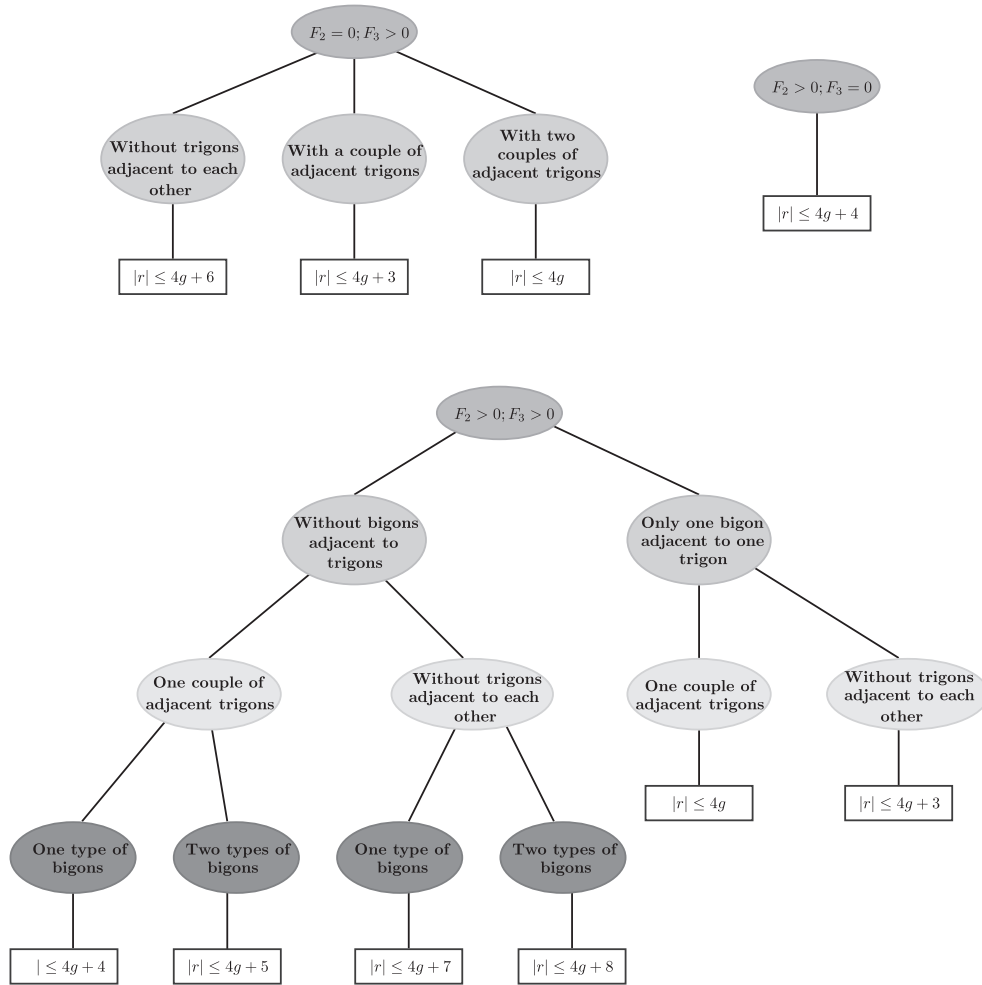


Fig. 17

disk face  $\sigma$  in  $G_S$  is adjacent to at most  $|\sigma| - 1$  bigons by Lemma 20. By counting the edges on  $G_S$  and applying (1), we have

$$\begin{aligned}
 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\
 &= -|F_3| + 4F - 3n \\
 &\geq 4(|r| - 2g(K) + 1) - 12
 \end{aligned}$$

Then,  $|r| \leq 4g(K) + 4$ .

### III. Without adjacent trigons and without bigons.

Since any two trigons are not adjacent and there is no bigons, the faces to which the trigons are adjacent have at least four sides. By counting the number of edges on  $G_S$  and applying (1), we have

$$\begin{aligned}
2|r| &\geq 3|F_3| + 4(F - |F_3| - m) + 3|F_3| \\
&= 2|F_3| + 4F - 4m \\
&\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 48
\end{aligned}$$

Then

$$(2) \quad |F_3| \leq 4g(K) + 22 - |r|$$

Also we have

$$\begin{aligned}
2|r| &\geq 3|F_3| + 4(F - |F_3|) \\
&= 4F - |F_3| \\
&\geq 4(|r| - 2g(K) + 1) - |F_3|
\end{aligned}$$

Then

$$(3) \quad |F_3| \geq -8g(K) + 4 + 2|r|$$

Comparing (2) and (3) gives  $|r| \leq 4g(K) + 6$ .

#### IV. Without bigons, with a couple of adjacent trigons.

By Lemma 26, if on  $G_S$  there is a couple of adjacent trigons, they are unique with respect to its type. Then  $G_S$  have this two trigons and other trigons of at most two types. Denote by  $F'_3$  the set of trigons that are not adjacent to any other trigon. Then  $|F_3| = |F'_3| + 2$ .

Let  $m'$  be the number of faces to which the trigons on  $F'_3$  are adjacent. By Lemmas 25 and 20,  $m' \leq 6$  and such faces have at least four sides. By counting the number of edges on  $G_S$  and applying (1), we have

$$\begin{aligned}
2|r| &\geq 3|F_3| + 4(F - |F_3| - m') + 3|F'_3| \\
&= 2|F_3| + 4F - 4m' - 6 \\
&\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 30
\end{aligned}$$

Then,

$$(4) \quad |F_3| \leq 4g(K) + 13 - |r|$$

Comparing (3) and (4) we get  $|r| \leq 4g(K) + 3$ .

#### V. Without bigons, with two couples of adjacent trigons.

By Lemma 26,  $G_S$  have only four trigons, i.e.  $|F_3| = 4$ . By counting the number of edges on  $G_S$  and applying (1), we get

$$\begin{aligned}
2|r| &\geq 3|F_3| + 4(F - |F_3|) \\
&= 4F - 4 \\
&\geq 4(|r| - 2g(K) + 1) - 4
\end{aligned}$$

Then,  $|r| \leq 4g(K)$ .

VI. One type of bigons, without bigons adjacent to trigons and with a couple of adjacent trigons.

By Lemma 14,  $G_S$  has at most three types of trigons. Since we do not have any bigon adjacent to any trignon, the faces to which the bigons are adjacent are at most two (Lemma 23) and have at least four sides.

On the other hand, we have a couple of adjacent trigons, then by Lemma 26, such trigons are the unique of its type. Since  $G_S$  has at most three types of trigons, then it has at most one type of trigons not adjacent to any trignon. Then we have  $n \leq 2$  and  $m \leq 3$ . By counting the number of edges on  $G_S$  and applying (1), we have

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| - 6 \\ &= 2|F_3| + 4F - 3n - 4m - 6 \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 24 \end{aligned}$$

Then

$$(5) \quad |F_3| \leq 4g(K) + 10 - |r|$$

Also

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 6 - |F_3| \end{aligned}$$

Then

$$(6) \quad |F_3| \geq -8g(K) - 2 + 2|r|$$

Comparing (5) and (6) we get  $|r| \leq 4g(K) + 4$ .

VII. Two types of bigons, one couple of adjacent trigons and without bigons adjacent to trigons.

Since any two bigons cannot be adjacent by Lemma 17, and we are assuming that any bigon is not adjacent to any trignon, then the faces to which the bigons are adjacent have at least four sides. On the other hand  $G_S$  has two types of bigons, then by Lemma 14 there are at most two types of trigons. We have also a couple of adjacent trigons, then by Lemma 26, they are the unique of its type and then  $|F_3| = 2$ . By counting the number of edges on  $G_S$  and applying (1), we have

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= -|F_3| + 4F - 3n \\ &\geq 4(|r| - 2g(K) + 1) - 14 \end{aligned}$$

Then

$$|r| \leq 4g(K) + 5.$$

VIII. One type of bigons, without bigons adjacent to trigons and without trigons adjacent between them.

By Lemma 14,  $G_S$  has at most three types of trigons. We are assuming that any bigon is not adjacent to any trignon, then the faces to which the bigons are adjacent have at least four sides.

Then  $n \leq 2$  and  $m \leq 9$ . By counting the number of edges on  $G_S$  and applying (1), we have

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| \\ &= 2|F_3| + 4F - 3n - 4m \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 42 \end{aligned}$$

Then

$$(7) \quad |F_3| \leq 4g(K) + 19 - |r|$$

Also

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 6 - |F_3| \end{aligned}$$

Then

$$(8) \quad |F_3| \geq -8g(K) - 2 + 2|r|$$

Comparing (7) and (8) we get that  $|r| \leq 4g(K) + 7$ .

IX. Two types of bigons, without bigons adjacent to trigons and without trigons adjacent between them.

By Lemma 14, on  $G_S$  there are at most two types of trigons. Then we have that  $n \leq 4$  and  $m \leq 6$ .

Since any bigon is not adjacent to any bigon or trignon, the faces to which the bigons are adjacent have at least four sides. The same happens to the trigons since they are not adjacent to any trignon. We also apply Lemma 20 to obtain

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n - m) + 2|F_2| + n + 3|F_3| \\ &= 2|F_3| + 4F - 3n - 4m \\ &\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 12 - 24 \end{aligned}$$

Then

$$(9) \quad 2|F_3| \leq 8g(K) + 32 - 2|r|$$

Also

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - n) + 2|F_2| + n \\ &= 4F - 3n - |F_3| \\ &\geq 4(|r| - 2g(K) + 1) - 12 - |F_3| \end{aligned}$$



Then

$$(10) \quad |F_3| \geq -8g(K) - 8 + 2|r|$$

Comparing (9) and (10) we get  $|r| \leq 4g(K) + 8$ .

X. With a bigon adjacent to a trigon and a couple of adjacent trigons.

By Lemma 18, there is only one bigon on  $G_S$ , i.e.  $|F_2| = 1$ . Also by Lemma 24, the trigon adjacent to the only bigon is the only one on its class, let  $t_1$  be such a trigon. We will see that  $|F_3| \leq 3$ .

Let  $t_2$  and  $t_3$  be two adjacent trigons on  $G_S$ . By Lemma 26,  $t_2$  and  $t_3$  are the unique with respect to its type. By Lemma 14,  $G_S$  has at most three types of trigons.

If  $t_1$ ,  $t_2$  and  $t_3$  were all different, they would be the only trigons on  $G_S$  and then  $|F_3| = 3$ .

Now suppose that two of  $\{t_1, t_2, t_3\}$  were equal, say  $t_1 = t_3$ . Let  $b$  be the only bigon on  $G_S$ . Suppose that  $b$  is black and it has class edges  $\{\chi_1, \chi_2\}$ . Then  $t_1$  is a  $\{\chi_2, \chi_3\}$ -white trigon and  $t_2$  is a  $\{\chi_3, \chi_4\}$ -black trigon ( $b$  and  $t_1$  are adjacent on an edge of class  $\chi_2$ , while  $t_1$  and  $t_2$  are adjacent on an edge of class  $\chi_3$ ). Applying repeatedly Lemma 16, we get a subgraph on  $G_S$  like Fig. 18.

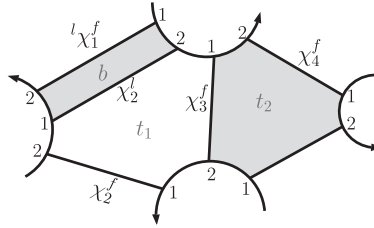


Fig. 18

If  $G_S$  had another type of trigons different than those already mentioned, it must be of type  $\{\chi_1, \chi_4\}$ -white trigon. But there is only one edge of class  $\chi_1$  on  $G_S$ , then there could be only one of such trigons. It follows that  $|F_3| \leq 3$ .

Therefore

$$\begin{aligned} 2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|) \\ &= 4F - |F_3| - 2|F_2| \geq 4(|r| - 2g(K) + 1) - 2 - 3 = 4|r| - 8g(K) - 1 \end{aligned}$$

Then

$$|r| \leq 4g(K).$$

XI. With a bigon adjacent to a trigon and without trigons adjacent between them.

Let  $b$  and  $t_1$  be a bigon and a trigon adjacent on  $G_S$ , respectively. By Lemmas 18 and 24,  $b$  is the only bigon on  $G_S$  and  $t_1$  is the only trigon of its type. By Lemma 14,  $G_S$  has at most three types of trigons. Then there is at most two types of trigons such that they are not adjacent to any trigon or bigon. Therefore  $m \leq 6$ . By counting the number of edges in  $G_S$  and applying (1) we have

$$\begin{aligned}
2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3| - m) + 3|F_3| - 3 \\
&= 2|F_3| + 4F - 4m - 5 \\
&\geq 2|F_3| + 4(|r| - 2g(K) + 1) - 29
\end{aligned}$$

Then

$$(11) \quad |F_3| \leq 4g(K) + 12 - |r|$$

Also

$$\begin{aligned}
2|r| &\geq 2|F_2| + 3|F_3| + 4(F - |F_2| - |F_3|) \\
&= 4F - 2|F_2| - |F_3| \\
&\geq 4(|r| - 2g(K) + 1) - 2 - |F_3|
\end{aligned}$$

Then

$$(12) \quad |F_3| \geq -8g(K) + 2 + 2|r|$$

Comparing (11) and (12) we get that  $|r| \leq 4g(K) + 3$ .

This concludes the proof.  $\square$

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